Exam 2017 Question

Zohaad Fazal

June 10, 2017

Sharpe-ratio estimator

Assume there are normally distributed log-returns which we don't observe.

$$X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$$

Instead, we observe whether the returns were higher than the risk-free rate r:

$$Y_i = 1_{\{X_i > r\}}$$

The Sharpe-ratio is defined as:

$$s = \frac{\mu - r}{\sigma}$$

We can define an estimator as:

$$\hat{s}_n = \Phi^{-1}(\bar{Y}_n),$$

where Φ^{-1} is the inverse of the standard normal CDF, $\bar{Y}_n$ the sample mean. The question is whether $\hat{s}_n$ is an unbiased estimator and whether it is consistent.

Analytic answer

Take $\Phi^{-1}(\Phi(\cdot))$ of both sides of the Sharpe-ratio:

$$\begin{aligned}
s &= \frac{\mu - r}{\sigma} \\
&= \Phi^{-1}(\Phi\left(\frac{\mu - r}{\sigma}\right)) \\
&= \Phi^{-1}(\mathbb{P}\left(\frac{X - \mu}{\sigma} < \frac{\mu - r}{\sigma}\right)) \\
&= \Phi^{-1}(\mathbb{P}\left(\frac{X - \mu}{\sigma} > -\frac{\mu - r}{\sigma}\right)) \\
&= \Phi^{-1}(\mathbb{P}(X > r)) \\
&= \Phi^{-1}(\bar{Y}_n)
\end{aligned}$$

The fourth line comes from the symmetry of the normal distribution. The conclusion is that $\hat{s}_n$ is an unbiased and consistent estimator.

R solution

```
set.seed(1234)
sim <- function(n, X.mu, X.sigma, rf) {
  X <- rnorm(n, X.mu, X.sigma)
  Y <- X > rf
```

```r
  sharpe.true <- (X.mu - rf) / X.sigma
  sharpe.est <-  qnorm(mean(Y), mean = 0, sd = 1) # inverse of standard normal CDF
  sharpe.true - sharpe.est
}

N = 10000
d <- sapply(1:N, FUN = function(i) {
  sim(n = 500, X.mu = 0.8, X.sigma = 1, rf = 0.25)
})

summary(d)
```

```
##       Min.    1st Qu.     Median       Mean    3rd Qu.       Max.
## -0.2221932 -0.0387932  0.0024487 -0.0003158  0.0427793  0.1915412
```

```r
# Fancy? Try #R-Finance at https://webchat.freenode.net/
```